CR 134560

Technical Report No. 108

The Effect of Stray Parameters on the Stability of a Class of Nonlinear Networks*

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This work was partially supported by the National Science Foundation under NSF GK 2283 and the National Aeronautics & Space Administration under Contract No. NGR 33-008-090.

*Presented at the 6th Annual Allerton Conference on Circuit & System Theory, September 1968.

THE EFFECT OF STRAY PARAMETERS ON THE STABILITY OF A CLASS OF NON-LINEAR NETWORKS

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Abstract: When analysing a nonlinear resistive network one normally considers its simplified resistive model which neglects the stray inductances and capacitances present in the network. The dynamics introduced by these stray elements may, in certain cases, critically affect the network response. For a certain class of resistive networks we show here that a certain lower bound on the ratio of the smallest stray capacitance to the largest stray loop inductance is sufficient for the global stability of the network.

I. INTRODUCTION

An important class of networks in current technology consists of those which are nonlinear and essentially resistive in nature. By this we mean that any reactive parameters occur unintentionally as stray or parasitic elements. Typical of these types of networks are those used in digital logic circuitry. For analysis purposes networks which are essentially resistive are normally approximated by resistive mathematical models, that is, models containing no dynamics. The analysis of such a model reduces to the determination of one or more equilibrium states. The physical network, however, contains certain dynamic effects due to the unavoidable presence of various stray reactive parameters, and these dynamic effects can lead to certain undesirable unstable behavior such as sustained parasitic oscillations. It is therefore of considerable practical importance to inquire into the conditions under which the response of an essentially resistive network from any initial state will, in spite of the stray reactances, remain bounded and tend toward an equilibrium state as $t \to \infty$. Under these circumstances the network will be said to possess a constant limiting regime (CLR). (A CLR implies the non-existence of any sustained oscillations).

The purpose of this paper is to delineate a class of essentially resistive networks in which sufficient conditions for a CLR can be derived and simply interpreted.

II. NETWORK DESCRIPTION

We shall limit the discussion to networks composed of elements whose resistive mathematical model satisfies the following conditions:

1) Each element has terminal relations which can be expressed in the form $i = f(v)$ where i, f and v are all n-vectors (column matrices) and f is continuously differentiable. Thus f represents a resistive n-port.

2) The function f is reciprocal [5] for each element.

3) For each element there exist positive constants N and ϵ such that* $\|v\| \geq N$ implies $v^t f(v) \geq \epsilon$

For any network element that is quasilinear (QL) [5], its terminal relations can also be written in the inverse form $v = f^{-1}(i)$. All elements that are not QL will be called voltage-controlled nonlinear resistors (VCNR). Note that any form of (reciprocal) nonlinear coupling is permitted among the various ports of an element. As will be apparent from our development, the dual case of networks containing QL resistors and current-controlled nonlinear resistors (CCNR) can be handled in an identical manner.

III. PROPER AUGMENTATION, NETWORK EQUATIONS

Since the actual distribution of stray reactive elements is not easy to determine, especially when the physical layout of the elements is unknown, we would like to obtain our results with as little information as possible concerning these parameters. The resistive network model will therefore be augmented by certain linear capacitances and inductances in a largely arbitrary fashion, subject to the following considerations.

It is known that for certain "irregular" nonlinear networks [1] the response of the mathematical model may be indeterminate. To avoid this possibility, we shall assume that each VCNR element has a positive capacitance of arbitrary value in parallel with each port. For the QL elements we shall assume that each port is either augmented by a series positive inductance or by a parallel positive capacitance (all of arbitrary values). (For simplicity, it will be assumed that all ports of the same element are augmented by the same type of reactance). Any augmentation consistent with the above restrictions will be said to be a "proper augmentation". In the sequel it will always be assumed that we are dealing with a properly augmented network.

To obtain the state variable equations for such a network we choose a "proper tree" following the procedure of Bryant [3] and write the fundamental loop and cutset matrices, following the notation of [5].

$$B = [I\ F] = \begin{bmatrix} I_{cc} & 0 & 0 & F_{cs} & 0 & 0 \\ 0 & I_{gg} & 0 & F_{gs} & F_{gr} & 0 \\ 0 & 0 & I_{\gamma\gamma} & F_{\gamma s} & F_{\gamma r} & F_{\gamma \ell} \end{bmatrix}$$

and $Q = [-F^t\ I]$

where $F_{gr} = 0$ because in the augmented network each resistive element has a capacitance in shunt or an inductance in series with each of its ports.

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* $\|\cdot\|$ denotes "norm" and $(\cdot)^t$ denotes "transpose".

The network differential equations will be [2]

$$\begin{bmatrix} \dot{\lambda} \\ \\ \dot{q} \end{bmatrix} = \begin{bmatrix} L & 0 \\ \\ 0 & C \end{bmatrix} \begin{bmatrix} \dot{i}_\gamma \\ \\ \dot{e}_s \end{bmatrix} = \begin{bmatrix} -F_{\gamma r} e_r & - F_{\gamma s}\, e_s \\ \\ +F^t_{gs} i_g & + F^t_{\gamma s}\, i_\gamma \end{bmatrix} \tag{1}$$

where $L = L_\gamma + F_{\gamma \ell} L_\ell F^t_{\gamma \ell}$

and $C = C_s + F^t_{cs} C_c F_{cs}$ are constant symmetric positive definite matrices, $i_g = g(e_g)$ is the current vector for the resistive elements included in the chords and e_r is the voltage vector for the resistive elements in the tree branches. Since only QL resistive elements may appear as tree branches, we can write $e_r = f^{-1}(i_r) = h(i_r)$.

The form of these equations is greately simplified if we define a scalar "dissipation function"

$$\mathfrak{J}(e_s, i_\gamma) = -\int g^t(-F_{gs} e_s)\, F_{gs}\, de_s - \int h^t\left(F^t_{\gamma r} i_\gamma\right) F^t_{\gamma r}\, di_\gamma - i^t_\gamma F_{\gamma s} e_s \tag{2}$$

where $\mathfrak{J}$ is assumed to be an indefinite line integral, (independent of path because of reciprocity [5]).

The network equations now become

$$\begin{bmatrix} L & 0 \\ \\ 0 & C \end{bmatrix} \begin{bmatrix} \dot{i}_\gamma \\ \\ \dot{e}_s \end{bmatrix} = \begin{bmatrix} (\partial \mathfrak{J}/\partial i_\gamma)^t \\ \\ -(\partial \mathfrak{J}/\partial e_s)^t \end{bmatrix} \tag{3}$$

IV. EXISTENCE OF A CONSTANT LIMITING REGIME

Based on Eq. (3) we can now derive a sufficient condition for the existence of a CLR. The following definitions will be required.

Let $R(i_\gamma) \triangleq F_{\gamma r} \dfrac{\partial h}{\partial i_r} F^t_{\gamma r}$

$G(e_g) \triangleq F^t_{gs} \dfrac{\partial g}{\partial e_g} F_{gs}$

If A is a real symmetric matrix, let

$$m(A) = \text{least eigenvalue of } A$$
$$M(A) = \text{greatest eigenvalue of } A$$

Let $$m_R = \text{Inf } m(R) \text{ over all } i_r \tag{4a}$$

$$m_G = \text{Inf } m(G) \text{ over all } e_g \tag{4b}$$

$$m_C = m(C) \tag{4c}$$

$$M_L = \frac{1}{2} M(L) \tag{4d}$$

Now, consider a properly augmented network N, described by equations (3) and satisfying the assumptions of Sect. II. In terms of the above quantities the following theorem may be stated:

Theorem: The network N will possess a constant limiting regime if

$$\frac{m_C}{M_L} > - \frac{2m_G}{m_R} \tag{5}$$

Before proceeding to the proof of the theorem, it is worthwhile to give a simple interpretation of condition (5) in terms of the structure of the network. It is easily shown that m_C is equal to or greater than the <u>smallest capacitor</u> in N, and M_L is equal to or less than the <u>largest loop inductance</u> in N, (i.e., the sum of all inductances in the loop [7]). Furthermore, m_R is derived from the incremental resistance matrix seen from the inductors when all capacitors are short-circuited. Because of the manner of augmentation, m_R must be positive.* On the other hand, m_G is related to the incremental conductance matrix seen from the capacitors (with inductors open-circuited), and may be negative.

Clearly if m_G were <u>positive</u>, condition (5) would automatically be fulfilled for <u>any</u> proper augmentation. In the more interesting case where m_G is negative the theorem indicates that <u>the ratio of smallest capacitance to largest loop inductance must be above a certain minimum value, $\frac{-2m_G}{m_R}$, in order to assure a CLR.</u> (This is, of course, a sufficient condition only).

Proof: We prove the theorem using two Lyapunov functions,

$$W = \frac{1}{2} e_s^t C e_s + \frac{1}{2} i_\gamma^t L i_\gamma \tag{6}$$

and

$$V = \frac{a\mathfrak{F}}{2} + \frac{1}{2} \left[\frac{\partial \mathfrak{F}}{\partial(i_\gamma, e_s)}\right]^t \begin{bmatrix} L^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} \left[\frac{\partial \mathfrak{F}}{\partial(i_\gamma, e_s)}\right] \tag{7}$$

* See proof of theorem.

The latter was first suggested by Brayton and Moser [6]. In (7), a is a positive constant to be determined subsequently. Using the direct method of Lyapunov [4], we can show the existence of a CLR if

1) $\dot{V} < 0$ everywhere except at the singular points of (3).

2) All trajectories are bounded for $t > 0$. [8, p. 66]

To prove condition (1), we take

$$\dot{V} = \begin{bmatrix} \dot{i}_\gamma \\ \\ \dot{e}_s \end{bmatrix}^t \left\{ \frac{a}{2} \begin{bmatrix} L & 0 \\ \\ 0 & C \end{bmatrix} + \begin{bmatrix} -R & F_{\gamma s} \\ \\ -F_{\gamma s}^t & -G \end{bmatrix} \right\} \begin{bmatrix} \dot{i}_\gamma \\ \\ \dot{e}_s \end{bmatrix}$$

$$= \dot{i}_\gamma^t \left(-R + \frac{a}{2} L\right) \dot{i}_\gamma + \dot{e}_s^t \left(-G - \frac{a}{2} C\right) \dot{e}_s \tag{8}$$

Since each VCNR is shunted by a capacitance, it must be a chord of a proper tree. Thus, all resistive tree branches are necessarily QL. Hence the function $h(i_r)$ must be QL in i_r. Since each inductive chord must be in series with a resistive element, the rank of the matrix $F_{\gamma r}$ will be equal to the number of rows of $F_{\gamma r}$. As a result the function $F_{\gamma r}h(F_{\gamma r}^t i_\gamma)$ will be QL in i_γ and the matrix R will be symmetric positive definite. [5, p. 36] No such conclusions can be drawn concerning the matrix G.

Now using definitions (4a-d) and substituting into (8) we may write

$$\dot{V} \le \dot{i}_\gamma^t \left(aM_L - m_R\right) \dot{i}_\gamma + \dot{e}_s^t \left(-m_G - \frac{am_C}{2}\right)\dot{e}_s$$

Assuming that condition (5) is fulfilled we can find an $a > 0$ satisfying,

$$\frac{m_R}{M_L} > a > \frac{-2m_G}{m_C}$$

Such a value of a will make $\dot{V} \le 0$ and $\dot{V} = 0$ only when $\dot{i}_\gamma = 0$ and $\dot{e}_s = 0$. Thus condition (1) is fulfilled.

To prove condition (2) we find from matrix B and eq. (1) that

$$\dot{W} = - e_g^t g(e_g) - i_r^t h(i_r)$$

It was assumed that $e_g^t g(e_g) \geq \epsilon > 0$ for large $\|e_g\|$ and $h(i_r)$ is a quasilinear function. Hence $W \to \infty$ and $\dot{W} < 0$ when $\|e_g\| + \|i_r\| \to \infty$. Due to the method of augmentation $\|i_\gamma\| + \|e_s\| \to \infty$ always implies $\|e_g\| + \|i_r\| \to \infty$. Thus using W as the Lyapunov function we can conclude that all solutions of eq. (1) will remain bounded.

This completes the proof of the theorem.

V. CONCLUSIONS

We have shown that by assuming physically reasonable distributions of stray reactive elements in a basically resistive network, certain sufficient conditions for a stable behavior can be derived. For a network composed of VCNR and QL resistive elements the conclusion to be drawn from these conditions is that stability will be assured if the stray inductances are sufficiently small compared to the stray capacitances. As a practical application of this result, consider the following situation. Suppose a network of the type discussed herein is constructed and is found to oscillate due to parasitic effects. Our results suggest that such oscillations can be suppressed by either reducing any stray inductances, or adding capacitances. (The latter may be easier to implement).

For the sake of clarity and brevity, many simplifications were made in the above development. Considerably sharper and more general results can be obtained by omitting these simplifications. For example, one can show that it is the smallest capacitance <u>in parallel with a VCNR</u> that determines the limit of stability, rather than the smallest capacitance in the network. Thus, to stabilize a network one should increase all VCNR capacitances.

Finally, we recall that all of the results can be dualized. Thus, for example, a network composed of CCNR's and QL resistances will be stable if its stray capacitances are sufficiently small compared to its stray inductances.

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